Economic Inquiry

TOURNAMENTS, FAIRNESS AND THE PROUHET-THUE-MORSE SEQUENCE

IGNACIO PALACIOS-HUERTA*

The Prouhet-Thue-Morse sequence occurs as the answer to various apparently unrelated questions in combinatorics, differential geometry, number theory, physics, music, turtle graphics, and other areas in the natural sciences. This paper shows that it can also be the answer to an important question in economics: how should the order of a sequential tournament competition between two agents be determined to make it fair? (*JEL* D00, Z00)

I. INTRODUCTION

Mathematician Axel Thue discovered in Thue (1912) the Prouhet-Thue-Morse sequence while studying avoidable patterns in binary sequences of symbols, e.g., 0, 1. This sequence is defined by forming the bitwise negation of the beginning:

$$\mathbf{t} = 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0$$

where 1 is the bitwise negation of 0, 1 0 is the bitwise negation of 0 1, 1 0 0 1 is the bitwise negation of 0 1 1 0, and so on. Formally, the Prouhet-Thue-Morse sequence $\mathbf{t} = (t_n)_{n \geq 0}$ is defined recursively by $t_0 = 0$ and $t_{2n} = t_n$, $t_{2n+1} = t'_n$ for all $n \geq 0$, where for $u \in \{0, 1\}$ we define $u' = 1 - u$.

This sequence $\mathbf{t}$ was already implicit in Prouhet (1851) and later rediscovered by Morse (1921) in connection with differential geometry. Worldwide interest in this sequence has developed during the past century as research has shown that it is ubiquitous in the scientific literature. In fact, this sequence occurs as a "natural" answer to various apparently unrelated questions, for instance, in combinatorics, in differential geometry, in number theory (e.g., the "Prouhet-Tarry-Escott" problem), in group theory (e.g., the Burnside problem), in real analysis (e.g., the Knopp function), in the physics literature on controlled disorder and quasi-crystals, in music, in chess, in fractals, and turtle graphics (e.g., the Koch snowflake), and in many others (see Allouche and Shallit 1999 for a survey).

Here we note that the Prouhet-Thue-Morse sequence can also be the natural answer to an important problem in economics: How should the order of a sequential tournament competition between two identical agents be determined to make it as fair as possible? That is, how should the order in a sequential tournament be determined to minimize any advantage that either competitor may have from both an ex ante perspective and an ex post perspective?

II. ANALYSIS

Tournament competitions are pervasive in real life and often characterize competitive situations such as competitions for promotion in internal labor markets in firms and organizations, patent races, political elections, student competitions in schools, penalty shoot-outs in soccer, chess matches, and many others (see Lazear and Rosen 1989; Prendergast 1999, and other references therein). There are numerous reasons why in a sequential tournament the order may give either a *first* mover advantage or a *second* mover advantage (see Dixit and Pindyck 1994; Cabral 2002, 2003, and other references therein). These include psychological reasons if, for instance, leading or lagging has an impact on the performance of the competing agents (Apesteguia and Palacios-Huerta 2010; González-Díaz and Palacios-Huerta 2011). In other words, a given order that is fair from an

*The author is Professor of Management at the London School of Economics and Ikerbasque (Basque Foundation for Science) scholar. I thank Thomas Stoll for letting me know about the various definitions and intriguing properties of the Prouhet-Thue-Morse sequence, Julio González-Díaz, Oscar Volij, and co-editor Yoram Bauman for useful comments and suggestions, and the Universidad del País Vasco UPV-EHU (Dept. Fundamentos del Análisis Económico I) for its hospitality.

Palacios-Huerta: Professor, Department of Management, London School of Economics, Houghton Street, London WC2A 2AE, UK. Phone +44-2071075167, Fax +44-2079556887, E-mail i.palacios-huerta@lse.ac.uk

Economic Inquiry
(ISSN 0095-2583)
Vol. 50, No. 3, July 2012, 848–849

doi:10.1111/j.1465-7295.2011.00435.x
Online Early publication January 6, 2012

ex ante perspective (e.g., randomly determined) need not be ex post fair if it gives any type of advantage (strategic or psychological) to either player.

Consider two players A and B that play against each other an even number of times. Typically a fair coin is used to randomly select the order of play, and the order follows a *strict alternation* of the chance to perform the task (e.g., in penalty shoot-outs in soccer, color of pieces in chess matches): A B A B A B A B A B A B . . . . The question we are interested in is the following: Is there a way to improve the ex post fairness of this sequential order?

Assume now that a coin flip selects A to perform his task first and B second in the first two rounds of a sequential competition. What should the order in the next two rounds be to make it ex post fair? If the order A B offers *any* kind of advantage to *either* player, then by reversing the order in the next two rounds we will tend to compensate for that advantage. The resulting sequence is: A B B A. And of course reversing is innocuous if no advantage existed in the first place. How about the next four rounds? The same principle applies: by reversing the order followed up to that point we will tend to compensate for any potential advantage that might have been given to either one of the players until then. The resulting sequence is: A B B A B A A B. And of course reversing is innocuous if no advantage existed, that is if A B B A in the first four rounds already provides no ex post advantage to either player.[1] Logically, by applying the same principle ad infinitum we obtain the Prouhet-Thue-Morse sequence:

A B B A B A A B B A A B A B B A . . .

III. DISCUSSION

Unfortunately, the Prouhet-Thue-Morse ordering is not followed in tournament competitions, including sports competitions, sequential auctions and others. The closest we find is serving in tie-breaks in tennis where the order of serves one and two (A B) is reversed for serves three and four (A B B A), and then this sequence is repeated A B B A A B B A A B B A . . . until a player wins by a certain margin.[2]

The Prouhet-Thue-Morse sequence, therefore, offers potential for improving the fairness of sequential tournament competitions.[3]

REFERENCES

Allouche, J.-P., and J. Shallit. "The Ubiquitous Prouhet-Thue-Morse Sequence," in *Sequences and Their Applications: Proceedings of SETA '98*, edited by C. Ding, T. Helleseth, and H. Niederreiter. London: Springer-Verlag, 1999, 1–16.

Apesteguia, J., and I. Palacios-Huerta. "Psychological Pressure in Competitive Environments: Evidence from a Randomized Natural Experiment." *American Economic Review*, 100(5), 2010, 2548–64.

Cabral, L. "Increasing Dominance with No Efficiency Effect." *Journal of Economic Theory*, 102(2), 2002, 471–79.

———. "R&D Competition When Firms Choose Variance." *Journal of Economics and Management Strategy*, 12(1), 2003, 139–50.

Dixit, A., and R. Pindyck. *Investment Under Uncertainty*. Princeton, NJ: Princeton University Press, 1994.

González-Díaz, G., and I. Palacios-Huerta. "Cognition and Emotion: Evidence from a Natural Experiment." Mimeo, LSE and Universidad de Santiago de Compostela, 2011.

Lazear, E., and S. Rosen. "Rank-Order Tournaments as Optimum Labor Contracts." *Journal of Political Economy*, 89(5), 1989, 841–64.

Morse, M. "Recurrent Geodesics on a Surface of Negative Curvature." *Transactions of the American Mathematical Society*, 22, 1921, 84–100.

Prendergast, C. "The Provision of Incentives in Firms." *Journal of Economic Literature*, 37(1), 1999, 7–63.

Prouhet, E. "Memoire sur Quelques Relations Entre les Puissances des Nombres." *Comptes Rendus de l'Académie des Sciences Paris Série I*, 33, 1851, 225.

Thue, A. "Uber unendliche Zeichenreihen," *Kra. Vidensk. Selsk. Skrifter. I. Mat.-Nat., Christiana*, Nr. 10. 1912. Reprinted in *Selected Mathematical Papers of Axel Thue,* edited by T. Nagell. Oslo: Universitetsforlaget, 1977, 139–58.

1. Let $\Delta(\mathbf{t}, n)$ denote the ex post difference in performance between the two identical subjects in a Prouhet-Thue-Morse sequence of 2^{n+1} elements, $n \geq 0$. Reversing will tend to compensate for any advantage if $|\Delta(\mathbf{t}, n)|$ decreases with n. A necessary and sufficient condition for the Prouhet-Thue-Morse sequence to be ex post fair is that $\lim_{n\to\infty} \Delta(\mathbf{t}, n) = 0$.

2. The serving order in tennis would be perfectly fair ex post if any advantage given by the order in the first two serves, A B, is exactly compensated for by having the order in the third and fourth serve reversed, B A. Of course, it is not known if this condition is empirically satisfied.

3. It is important that the sequence has 2^{n+1} elements, $n \geq 0$, that is, that its first half is the negation of the second half. Otherwise the full potential will not be realized (e.g., in a soccer penalty shoot-out the winner should be the best of $2^3 = 8$ penalty kicks or best of $2^4 = 16$, etc., not the best of 10 as it currently is). The margin of victory chosen to determine the winner is irrelevant for the ex post fairness of a sequence with 2^{n+1} elements.